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Dynamics of discrete breathers in the integrable model of the 1D crystal

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In the frame of the exactly integrable model of the 1D crystal — Hirota lattice model — the dynamics and interaction of the discrete breathers has been considered. These high-frequency localized nonlinear excitations elastically interact with each other and with such excitations as shock and linear waves. Using the nonlinear superposition formula the pair collision processes of the excitations are analytically described and explicit expressions for center-of-mass shifts of shock waves (kinks) and breathers, and phase shifts of oscillations of breathers and linear waves are discussed. The dynamics of the discrete breathers and kinks as the particle-like excitations of the Hirota lattice is described using the Hamiltonian formalism. The exact nonlinear periodic solutions describing the breathers and solitons superlattices in the Hirota lattice are analysed, and their stability boundaries are determined. The analogue of the discrete breather for the finite-size system is presented in terms of the elliptic Jacobi functions and it is shown that the excitation is detached from the branch of nonlinear homogenous antiphase oscillations in the bifurcation manner.

Keywords: discrete breather, intergable models, soliton interaction, breather superlattices.

1. The 1D Hirota lattice model

The exactly integrable models of the lattice systems are of significant importance for the solid state physics [1]. The equations describing these models have the exact solutions allowing to investigate analytically the dynamics of the lattice system and it is possible to obtain the analytic expressions for the main physical characteristics of the excitations such as energy, momentum, *etc*. The most known integrable lattices are the Toda lattice [2] and the system of Ablowitz-Ladik [3] which is the integrable analog of the discrete nonlinear Schrödinger equation. The Toda lattice equation describes the 1D anharmonic crystal with the exponential interaction force between the nearest neighbors. The Ablowitz-Ladik equation describes the system of parallel nonlinear optical waveguides but also it is widely used in the other areas because it has the exact integrable quantum analog [4]. In 1973 Hirota [5] suggested the exactly integrable system of the nonlinear self-dual network (NSDN) equations for currents strengths and voltages describing the transmission line with the nonlinear inductances and capacitances. The NSDN equations are the nonlinear telegraph equations.

The mechanical analog of the nonlinear transmission line is a one-dimensional anharmonic chain of atoms, for which only the nearest neighbors interaction is taken into account [6,7].

$$\frac{m\ddot{u}_n}{1+\frac{\pi^2}{4}\frac{\dot{u}_n^2}{s^2}}=\frac{2}{\pi}\gamma d_0(\Delta_{n-1,n}-\Delta_{n,n+1}), \tag{1}$$

where $\Delta_{n,n+1}=\tan[(\pi/2)(u_n-u_{n+1})/d_0]$ and u_n is the displacement of the n-th atom in the chain. In the left part of (1) there is a kinetic term similar as for the modified discrete sine-Gordon model [8]. In the right side there are the nonlinear (tangential) interaction forces between the nearest neighbors. Equation (1) is called the Hirota lattice equation [6,7]. The Lagrangian function corresponding to the Eq.(1) has the form [6,7]:

$$L=\frac{4}{\pi^2}ms^2\sum_{n=-\infty}^{+\infty}L_n \tag{2}$$

$$L_n=\frac{\pi}{2s}\dot{u}_n\arctan\left(\frac{\pi}{2s}\dot{u}_n\right)-\\-\frac{1}{2}\ln\left[1+\left(\frac{\pi}{2s}\dot{u}_n\right)^2\right]-\frac{1}{2}\ln\left[1+\Delta_{n-1,n}^2\right] \tag{3}$$

For small values of the displacements and velocities of atoms the Hirota lattice model reduces to the β-Fermi-Pasta-Ulam (β-FPU) lattice model [9]. In [10] using the limiting phase trajectories concept and the notion of effective particles it was revealed the origin of transition from energy exchange to energy localization and transfer in finite periodic FPU chains. Equation (1) is equivalent to the discrete modified Korteweg – de Vries (dmKdV) equation [5]. In the long-wave limit Eq.(1) reduces to the continuous modified Korteweg – de Vries (mKdV) equation [5]. Hirota [5] has shown that the system of NSDN equations and Eq.(1) are exactly integrable and has found their multi-soliton solutions. Ablowitz and Ladik [11] have found the soliton solutions and the conserved quantities of the system of NSDN equations. Bogdan [6] obtained the moving and standing discrete breather (DB) solutions for the Eq.(1) for the first time. In the dimensionless coordinates

$$\frac{u_n}{2d_0/\pi} = \phi_n, \quad \frac{\tau}{\left(d_0/s\right)} = t \tag{4}$$

the DB solution of the Eq.(1) has the form:

$$\phi_n^{(b)}(t) \equiv \phi\left(X_b, \Phi\right) = \\ = \arctan\left[\frac{\sinh(\kappa/2)}{\sin(k/2)} \frac{\cos(\Phi)}{\cosh\kappa\left(n - X_b\right)}\right], \tag{5}$$

where the center-of-mass coordinate and the phase of oscillations have the form:

$$X_b = Vt + X_{b0}, \quad \Phi = k(n - X_b) - (\omega - kV)t + \Phi_0 \tag{6}$$

center-of-mass velocity and cyclic frequency of oscillations equal

$$V = \pm(2/k)\sinh(k/2)\cos(k/2), \quad \omega = \pm 2\cosh(k/2)\sin(k/2) \tag{7}$$

The DB solution of the Eq.(1) has the similar structure as the breather of sine-Gordon equation. Zhou et al. [12] have obtained the discrete breather solution (5) of the Eq.(1) using the wronskian technique.

Discrete breathers (DB) or intrinsic localized modes (ILM) are the spatially localized and periodic in time nonlinear excitations of the lattice systems [13,14].

During the last decade DBs have been investigated intensively in condensed matter physics and materials science. In the review [15] the results on gap DBs in two- and three-dimensional crystals have been summarized. In [16] the molecular-dynamics simulations of DBs in the crystals with NaCl structure with different ratios of atomic masses of components have been presented. In [17] the DBs, many-frequency breathers as DBs of a new type and quasibreathers in nonlinear monoatomic chains have been analyzed. In [17] a general method for constructing DBs which provides the pair synchronization between the individual particles` vibrations is discussed.

2. The nonlinear superposition formula for the Hirota lattice model. The processes of the DBs and shock waves pair collisions.

In [18] the nonlinear superposition formula has been found for the Eq.(1) in the dimensionless coordinates (4). It is a recurrence relation allowing to generate more complicated soliton solutions using more simple soliton solutions. The nonlinear superposition formula is as follows:

$$\phi^{(N+1)} = \phi^{(N-1)} + \arctan\left[\frac{A_i^{\varepsilon_i\varepsilon_j} + A_j}{A_i^{\varepsilon_i\varepsilon_j} - A_j}\tan\left(\phi_I^{(N)} - \phi_{II}^{(N)}\right)\right], \\ A_i = \tanh\left(K_i/4\right) \tag{8}$$

where K_i is the parameter of the i-th soliton, $\varepsilon_i = \pm 1$ corresponds to the sign of the i-th soliton velocity.

The multi-soliton solutions of larger order can be constructed in series beginning from the trivial and one-soliton solutions. One has to substitute two soliton solutions of the N-th order, which have different values of the parameter K_i. The formula (6) connects four soliton solutions of different order–one solution of the (N-1)-th order, two solutions of the N-th order and one solution of the (N+1)-th order, *e.g.* one two-soliton solution, two three-soliton solutions and one four-soliton solution.

A similar formula is known for the sine-Gordon [19,1] and mKdV [20] equations. In [18] the received nonlinear superposition formula has been extended to the case of breathers. It was shown how using the nonlinear superposition formula to derive the breather and wobbling kink [21] solutions. For the Eq.(1) the wobbling kink solution has been derived for the first time in [18].

Schematically the procedure of constructing the soliton, breather and wobbling kink solutions with the use of the nonlinear superposition formula can be depicted by means of the Lamb diagram (Fig.1).

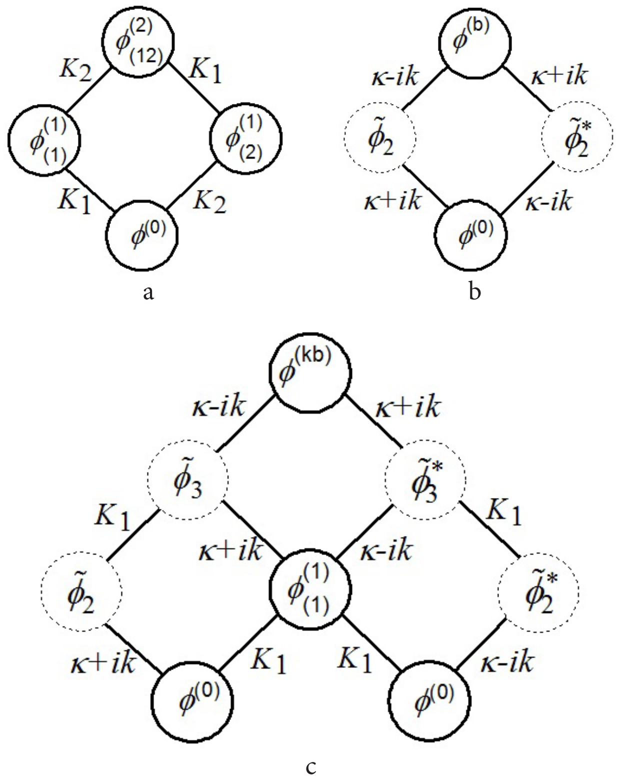

Fig. 1. The Lamb diagram for constructing the 2-soliton solution (a), the breather solution (b) and the wobbling kink solution (c) of the Hirota lattice equation using the nonlinear superposition formula. The wobbling kink solution is obtained from the "kink-breather" solution by choosing the definite values of the parameters. The asterisk denotes the complex conjugate.

To obtain breather solution one has to take the complex conjugate numbers as the parameters of two solitons (Fig.1b,c).

It is known that the interaction of the arbitrary number of solitons can be described by the sequence of their pair collisions. Currie [22] has extended the method of soliton collision analysis for the breather solutions in continuous systems. So it became possible to study the interaction of discrete breathers with each other and with the other excitations of the system (kinks and linear waves). In [22—24] the interaction of breathers, kinks and linear waves is investigated for the sine-Gordon equation.

In [25] for the exactly integrable Hirota lattice model and equivalent NSDN and dmKdV equations the pair collision processes of two discrete breathers, breather and one-parametric soliton (kink, antikink), breather and linear wave, one-parametric soliton and linear wave are described. The explicit expressions of the "kink-breather" and "breather-breather" solutions are constructed. The shifts of the center-of-masses and phases of the breather oscillations have been expressed in terms of the dynamical characteristics of linear and nonlinear excitations of the system.

In general case for the arbitrary values of the parameters of the "kink-breather" solution the shifts of the center-of-mass of one-parametric soliton (kink or antikink) and breather and the shift of breather phase of the oscillations have the following form:

$$\Delta X_1 = -\mathrm{sign}\left[k\left(V_1 - V_b\right)\right]\frac{2\ln R}{K_1}, \qquad \Delta X_b = -\mathrm{sign}\left[K_1\left(V_b - V_1\right)\right]\frac{\ln R}{k} \tag{9}$$

$$\Delta\Phi = +\mathrm{sign}[K_1(V_b - V_1)]\delta + \pi \tag{10}$$

For the arbitrary values of the parameters of the "breather-breather" solution the shifts of the breathers' center-of-masses and phases of the oscillations have the form:

$$\Delta X_i = -\mathrm{sign}\left[\kappa_j\left(V_i - V_j\right)\right]\frac{\ln R_{12}R_{14}}{\kappa_i}, \qquad \Delta\Phi_i = +\mathrm{sign}\left[\kappa_j\left(V_i - V_j\right)\right]b_i \tag{11}$$

where $i,j = 1,2$; $i \neq j$. The expressions for the parameters R, δ, R_{12}, R_{14}, b_1, b_2 are complicated, they can be found in [25]. The special cases of collision processes have been discussed in [25]: the interaction of kink and standing breather; the interaction of kink and linear wave; the interaction of moving and standing discrete breather, the interaction of discrete breather and linear wave, the linear superposition of two linear waves. The analysis of the scattering data [25] has shown that the DBs and shock waves of the Hirota lattice model interact with each other by means of the effective short-range forces of attraction.

Results obtained in [25] can be used for the quantitative description of the discrete one-parametric soliton and breather propagation and reflection from the free and fixed boundaries in the lattice and in constructing the low-temperature thermodynamics.

3. The Hamiltonian dynamics of the DBs and shock waves of the Hirota lattice model.

The collective coordinate method gives the possibility to describe the dynamics of solitons in the nonlinear lattices as particle-like excitations [3]. The problem of the Hamiltonian dynamics of the DBs of the Ablowitz-Ladik equation was considered in [26].

In [27] the formula for constructing the analogs of the field pulses for the discrete kinks and breathers has been presented. F. G. Mertens and H. Büttner used "angle-action" variables to describe the dynamics of the solitons in the Toda lattice [28]. In [27] the formula for the generalized momentum of the excitation in the chain of atoms conjugated to the generalized coordinate q_i was proposed:

$$P_{q_i} = \frac{\oint dq_i\left(\sum_{n=-\infty}^{+\infty}\frac{\partial L}{\partial \dot{u}_n}\frac{\partial u_n}{\partial q_i}\right)}{\oint dq_i} \tag{12}$$

where the sign $\oint$ denotes the integration over the complete change of the coordinate [29]. Formally the choice of such generalized variables corresponds to the transition to the variables "angle-action", where angle is the definite generalized coordinate q_i, and the action is the corresponding to it generalized momentum P_{qi}.

In [6,7] the basic physical integrals of motion: energy, center-of-mass pulse, energy flow have been found for kinks and breathers of the Hirota lattice model, as well as the adiabatic invariant for breather. The energy spectrum of the kink $E_1(P_{X1})$

$$E_1 = \frac{K_1}{2}, \qquad P_{X_1} = \int_0^{K_1/2}\frac{x}{\sinh x}dx \tag{13}$$

and the quasi-classical energy spectrum of breather $E_b = E_b(P_{Xb}, P_\Phi)$

$$E_b = \kappa, \quad P_{X_b} = 2\,\mathrm{Re}\left(\int_0^{(\kappa+ik)/2}\frac{x}{\sinh x}dx\right), \qquad P_\Phi = \arctan\left[\frac{\sinh(\kappa/2)}{\sin(k/2)}\right] \tag{14}$$

have been obtained [7].

From (13) the expression between the variations of kink energy, center-of-mass pulse and velocity was derived

$$\delta E_1 = V_1 \delta P_{X1}. \tag{15}$$

From (15) one can obtain the canonical Hamilton equations describing the uniform rectilinear motion of the kink center-of-mass along the atomic chain.

From (14) the following equation was derived:

$$\delta E_b = V\delta P_{Xb} + \omega'\delta P_\Phi \tag{16}$$

From (16) one can obtain two pairs of canonical Hamilton equations describing the uniform rectilinear motion of the breather center-of-mass along the atomic chain with velocity V and the breather oscillations with the constant frequency ω' in the reference frame moving with the velocity of the breather center-of-mass respectively.

In the long-wave limit the dependence of DB energy on generalized pulses has the form:

$$\frac{\tilde{E}_b\left(P_{X_b},P_\Phi\right)}{ms^2}=\frac{P_{X_b}}{ms}+\frac{1}{72}\left(\frac{\pi}{2}\right)^4\cdot$$
$$\cdot\left(\frac{P_{X_b}}{ms}\right)^3\left[1-3\cot^2\left(\left(\frac{\pi}{2}\right)^2\frac{P_\Phi}{msd_0}\right)\right] \qquad (17)$$

In the quasi-classical limit the value of action P_Φ, which play the role of the generalized pulse of the breather for oscillating degree of freedom, is quantized. It equals the integer number N of the action quantum $\hbar$. For large $N>>1$ the following expression is valid

$$P_\Phi=\hbar(N+const)\approx\hbar N. \qquad (18)$$

The expression (17) with (18) is the energy spectrum of the DB of the Hirota lattice model in the long-wave quasi-classical limit.

From (17) and (18) the following equation can be obtained:

$$\partial\tilde{E}_b/\partial N=\hbar\omega'=\hbar(\omega-kV). \qquad (19)$$

The models of the 1D ideal anharmonic crystal and equivalent nonlinear transmission line which have been proposed by Hirota were generalized by adding the terms corresponding to the dissipation processes and the action of the external forces [30]. The equation for the generalized 1D Hirota lattice model has the form:

$$\frac{m\ddot{u}_n}{1+\frac{\pi^2}{4}\frac{\dot{u}_n^2}{s^2}}=\frac{2\gamma d_0}{\pi}(\Delta_{n-1,n}-\Delta_{n,n+1})-$$
$$-\frac{\pi}{2}\lambda\frac{\dot{u}_n}{s}+F^{(ext)}(t) \qquad (20)$$

where $F^{(ext)}(t)$ is the external varying in time homogeneous force. For the Eq.(20) the dissipation function describing the energy dissipation was introduced [27].

The generalized system of the NSDN equations describing the equivalent nonlinear transmission line in dimensionless units has the form [30]:

$$\frac{d}{dt}\left[C(V_n)V_n\right]+GV_n=I_n-I_{n+1},$$
$$\frac{d}{dt}\left[L(I_n)I_n\right]+RI_n=V_{n-1}-V_n+E(t) \qquad (21)$$

where:

$$C(V_n)=V_n^{-1}\arctan V_n,\quad L(I_n)=I_n^{-1}\arctan I_n. \qquad (22)$$

In [31] the periodic vibrations that represent the symmetry-determined nonlinear normal modes have been investigated using the group-theoretical method in the LC- and LCR-transmission lines.

4. The highly discrete and long-wave limits of the Hirota lattice model

In this section the highly discrete and long-wave limits of the Hirota lattice model are considered. In the highly discrete limit the excitation is localised practicaly on the one site of the lattice. This limit can be qualitatively described using the model of the anharmonic Hirota oscillator.

In [27] two models of the Hirota nonlinear oscillator are investigated – 1D oscillator with one spring

$$\ddot{\phi}\Big/\left(1+\dot{\phi}^2\right)=-\tan\phi \qquad (23)$$

and 1D oscillator with two springs from both sides

$$\ddot{\phi}\Big/\left(1+\dot{\phi}^2\right)=-2\tan\phi. \qquad (24)$$

The corresponding equations can be derived from (1) if we consider only one lattice site and if we assume that there is interaction only with one nearest neighboring atom or with two nearest neighboring atoms, with one from each side.

For the model of the Hirota oscillator with one spring the exact solution describing the oscillations of single atom has been found:

$$\phi=\arcsin\left[\chi\,\mathrm{sn}\left(|A_1|t,\chi\right)\right],\quad |A_1|=\left(1-\chi^2\right)^{-1/2}, \qquad (25)$$

where $\mathrm{sn}(\theta,\chi)$ is the Jacobi elliptic sine function. It is easy to see that the cyclic frequency of oscillator has the form

$$\omega=\pi\Big/\left(2\mathbb{K}(\chi)\right)\left(1-\chi^2\right)^{-1/2}, \qquad (26)$$

where $\mathbb{K}(\chi)$ is the complete elliptic integral of the first kind, χ is the modulus of the Jacobi elliptic functions. The energy spectrum of the oscillator in classical case has the form [27]:

$$E=-\frac{1}{2}\ln\left(1-\chi^2\right),\quad P_\Phi=\frac{2}{\pi}\int_0^\chi\frac{\chi}{\sqrt{1-\chi^2}}\mathbb{K}(\chi)\,d\chi \qquad (27)$$

From (27) and (26) the standard ratio for the oscillators follows

$$dE/dN=\omega. \qquad (28)$$

The generalized momentum P_Φ is the action. From quantum mechanics [32] it is known that in the quasi-classical limit the Bohr-Sommerfeld rule is valid. In the limit of large values of the principal quantum number the Eq.(18) is valid. Equations (27) with (18) represent the energy spectrum of the Hirota oscillator with one spring in the quasi-classical limit.

The exact solution and energy spectrum for the model of Hirota oscillator with two springs have been obtained [27] using the similar calculations as for the one spring model.

It was shown [27] that after the change of variables:

$$\phi=\arctan(f) \qquad (29)$$

the model of Hirota oscillator with two springs is reduced to the model of the Duffing oscillator with positive coefficients:

$$\ddot{f}+2f+2f^3=0 \qquad (30)$$

In literature the classical model of the nonlinear Duffing oscillator is well-known. [33] L.-Z. Guo et all [34] have investigated the nonlinear dynamics of a mesoscopic driven Duffing oscillator in a quantum regime.

Consider the long-wave and small-amplitude limit of the Hirota lattice model. In this limit the characteristic width of the localized excitation is much larger then lattice constant and the discreteness of the system can be neglected. After transition to the frame of reference moving with the speed of sound and introducing the slow time

$$\xi = \varepsilon(x - st), \quad \tau = \varepsilon^3 t, \tag{31}$$

where ε<<1 is the small parameter, Eq.(1) reduces at first to the modified Boussinesq and then to the mKdV Eq.(32):

$$2msu_{\xi\tau} + \frac{\gamma d_0^4}{12} u_{\xi\xi\xi\xi} + \frac{\gamma d_0^2 \pi^2}{2} (u_\xi)^2 u_{\xi\xi} = 0. \tag{32}$$

In the small-amplitude and long-wave limit and after the change of the vaiables *(x,t)* → *(ξ,τ)* the integrals of motion of the Hirota lattice model can be expressed in terms of the integrals of motion of the mKdV equation:

$$E \to \varepsilon s P_X^{mKdV} + \varepsilon^3 E^{mKdV} - \varepsilon^3 I + o(\varepsilon^3), \tag{33}$$

$$P_X \to \varepsilon P_X^{mKdV} - \varepsilon^3 I + o(\varepsilon^3), \quad P_\Phi \to \varepsilon^3 P_\Phi^{mKdV} + o(\varepsilon^3) \tag{34}$$

where:

$$I = \frac{\gamma d_0^4}{24} \int_{-\infty}^{+\infty} d\frac{\xi}{d_0} (u_{\xi\xi})^2. \tag{35}$$

For the kink and breather solutions of the mKdV equation the expressions for soliton energy and pulses have been obtained. The dependence of energy on center-of-mass pulse for the kink has the form

$$E_1^{mKdV} = sP_{X_1}^{mKdV} + ms^2 \frac{1}{18}\left(\frac{\pi}{2}\right)^4 \left(\frac{P_{X_1}^{mKdV}}{ms}\right)^3, \tag{36}$$

which corresponds to the limit of small pulses of the dependence (13). From (36) the Hamiltonian equations for kink can be easily derived.

The dependence of breather energy of the mKdV equation on pulses $P_{X_b}^{mKdV}, P_\Phi^{mKdV}$ has the same form as the breather excitation spectra (17) of the Hirota lattice model in the long-wave limit. The energy of the mKdV breather is less then energy of two free one-parametric solitons.

$$E_b = 2E_1 - \Delta E,$$
$$\Delta E = \pi^4 P_{X_b}^3 / (384 m^2 s) \cot^2\left[\pi^2 P_\Phi / (4msd_0)\right], \tag{37}$$

It is obvious that $\Delta E=0$ for

$$P_\Phi = P^*_{\Phi,l} = 2msd_0(2l-1)/\pi, \; l=1,2,3,... \tag{38}$$

In this case breather breaks down into kink and antikink. In the quasi-classical limit the energy spectrum of the mKdV breather has the form (17), (18).

5. The superlattices of the discrete breathers and shock waves

In [30] the new classes of periodic solutions expressed in terms of the Jacobi elliptic functions have been obtained for the Hirota lattice model and equivalent system of NSDN equations.

The obtained solutions are the spatially periodic waves describing the discrete breather and shock wave superlattices. In the small-amplitude limit these solutions reduce to the linear running and standing waves and in the essentially nonlinear limit to the separated discrete breathers or one-parametric solitons (kinks and antikinks). The new solutions have been found for the infinite lattice, however they can also satisfy the periodic and zero-fixed boundary conditions for the finite-size lattices.

Kovalev [35] has obtained the cnoidal waves solutions of the sine-Gordon equation. In [36] the breather lattice solution of the sine-Gordon equation has been investigated. In [37] the exact periodic solutions of the positive and negative modified Korteweg-de Vries equations have been found.

For brevity we will use the following notation

$$s_1 \equiv \mathrm{sn}(a/2,\chi), c_1 \equiv \mathrm{cn}(a/2,\chi), d_1 \equiv \mathrm{dn}(a/2,\chi),$$
$$s_2 \equiv \mathrm{sn}(p/2,m), c_2 \equiv \mathrm{cn}(p/2,m), d_2 \equiv \mathrm{dn}(p/2,m).$$
$$\chi' = \sqrt{1-\chi^2}, \qquad m' = \sqrt{1-m^2}. \tag{40}$$

where sn(θ,χ), cn(θ,χ), dn(θ,χ) are the Jacobi elliptic functions, χ, *m* are the modules of the elliptic functions, χ', *m*' are the additional modules of the elliptic functions.

In [30] the new solution of the Eq.(1) has been found for the first time

$$\phi_n(t) = \arctan\left[A\,\mathrm{sn}(an+bt,\chi)\,\mathrm{dn}(pn+qt,m)\right]. \tag{41}$$

where

$$A = \sqrt{\frac{\chi}{m'}} = \frac{s_2}{s_1 c_2}. \tag{42}$$

$$b = \pm\frac{2s_1 d_2}{1 - A^2 s_2^2 d_2^2}, \quad q = \pm\frac{2s_2 c_1 c_2 d_1}{1 - A^2 s_1^2 d_2^2}, \tag{43}$$

Solution (41) describes the moving discrete breather superlattice of type I.

Using the periodicity property of the Jacobi elliptic functions the expression describing the nonlinear inhomogeneous antiphase oscillations in the chain of atoms was obtained from the Eq.(41).

$$\phi_n(t) = (-1)^n \arctan\left[A\,\mathrm{sn}(bt,\chi)\,\mathrm{dn}(pn,m)\right]. \tag{44}$$

For $m\to 0$ expression (44) reduces to the solution describing the nonlinear homogeneous antiphase oscillations:

$$\phi_n(t) = (-1)^n \arctan\left[\sqrt{\chi}\,\mathrm{sn}\left(\frac{2}{1-\chi}t,\chi\right)\right]. \tag{45}$$

In Fig.2 the dependence of the cyclic frequency on energy for the homogeneous and inhomogeneous oscillations of the chain with the number of nodes *N*=6 is shown. The number of spatial periods that fit the length of the chain is *M*=1. Dashed line shows the dependence of discrete breather frequency on energy for the infinite chain. The boundary conditions are

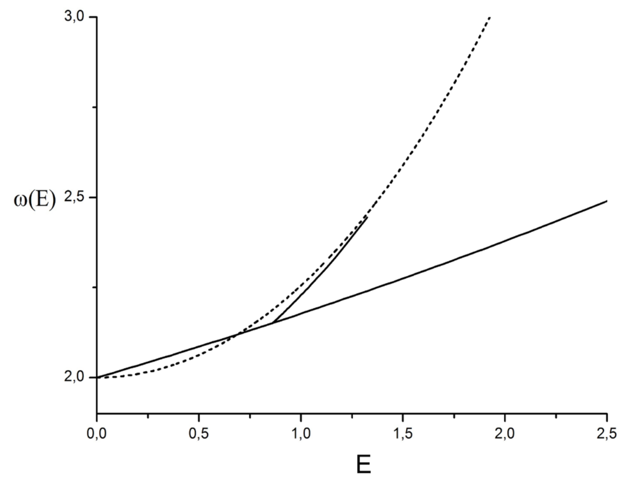

Fig. 2. The dependence of frequency on energy $\omega=\omega(E)$ for the homogeneous and inhomogeneous oscillations of the chain with the number of nodes N=6, M=1. The boundary conditions are periodic. Dashed line shows the dependence of discrete breather frequency on energy for the infinite chain.

periodic.

The brunch corresponding to the inhomogeneous oscillation is detached from the brunch corresponding to the homogeneous oscillation by the bifurcation way and is pressed against the breather branch (dashed line).

From the Fig.2 it is seen that the inhomogeneous oscillation is energetically more favorable then the corresponding homogeneous oscillation of the same frequency.

To examine the dynamical stability of the obtained solutions the direct numerical simulations were used. The time integration has been performed by means of a 7th [30] order Runge-Kutta scheme. The space-time evolution of the breather lattice solution of the type I for the ideal and dissipative lattice was investigated. The periodic boundary conditions have been used. Simulations demonstrate the dynamical metastability of the solution in the ideal lattice and instability in the dissipative lattice.

In [30] the solution describing the moving discrete breather superlattice of type II has been found.

$$\phi_n(t) = \arctan\left[A\mathrm{cn}(an+bt,\chi)\,\mathrm{cn}(pn+qt,m)\right] \quad (46)$$

where:

$$b = \pm\frac{2s_1c_2d_1\left(1+A^2\right)}{1+A^2c_1^2c_2^2},\quad q = \pm\frac{2s_2c_1d_2\left(1+A^2\right)}{1+A^2c_1^2c_2^2}. \quad (47)$$

$$A = \sqrt{\frac{\chi m}{\chi' m'}} = \sqrt{\frac{d_2^2-d_1^2}{c_1^2d_2^2-c_2^2d_1^2}}. \quad (48)$$

Using the periodicity property of the Jacobi elliptic functions the expression describing the nonlinear inhomogeneous antiphase oscillations in the chain of atoms was obtained from the Eq.(46).

$$\phi_n(t) = (-1)^n \arctan\left[A\mathrm{cn}(an,\chi)\,\mathrm{cn}(qt,m)\right]. \quad (49)$$

In Fig.3 the space-time evolution of the breather superlattice solution of the type II for the ideal (Fig.3a), dissipative with λ=0.1 (Fig.3b) and driven-damped with λ=0.1, F_i=0.5cos(ωt), i=5,10,15,20,25,30 (Fig.3c) lattice is shown. The external forces are applied to the center-of-mass of each breather in the superlattice. The frequency of the external force equals to the frequency of the breather superlattice solution of type II. The periodic boundary conditions have been used. Simulations demonstrate the stability of the solution in the ideal lattice and instability in the dissipative lattice. The lifetime of breather superlattice solution of the type II in the dissipative lattice can be extended through the concurrent application of ac driving and viscous damping terms.

In [38] the results of the inelastic neutron measurements performed on the NaI crystals show the existence of the discrete breather superlattice under certain conditions.

In [30] the solution describing the moving discrete shock wave superlattice has been found.

$$\phi_n(t) = \arctan\left[A\mathrm{sc}(an+bt,\chi)\,\mathrm{dn}(pn+qt,m)\right], \quad (50)$$

where:

$$b = \pm\frac{2s_2c_2d_1}{s_2^2d_1^2-A^2c_2^2},\quad q = \pm\frac{\chi^2}{m^2}\frac{2s_1c_1d_2}{s_2^2d_1^2-A^2c_2^2}. \quad (51)$$

$$A = \sqrt{\frac{\chi'}{m'}} = \frac{d_1}{d_2}. \quad (52)$$

$$\mathrm{sc}(\theta,\chi) = \mathrm{sn}(\theta,\chi)/\mathrm{cn}(\theta,\chi). \quad (53)$$

Using the periodicity property of the Jacobi elliptic functions the expressions describing the soliton-soliton (54) and soliton-antisoliton (55) superlattices in the chain of atoms were obtained from the Eq.(50).

$$\phi_n(t) = \arctan\left[A\mathrm{sc}(an,\chi)\,\mathrm{dn}(qt,m)\right]. \quad (54)$$

$$\phi_n(t) = \arctan\left[A\mathrm{sc}(bt,\chi)\,\mathrm{dn}(pn,m)\right]. \quad (55)$$

The space-time evolution of the soliton-soliton and soliton-antisoliton superlattice solutions for the ideal and dissipative lattice was investigated. The periodic boundary conditions have been used. Simulations [30] demonstrate the stability of the solution in the ideal lattice and instability in the dissipative lattice.

Conclusions

In this review the dynamics and interaction of the discrete breathers in exactly integrable model of the 1D anharmonic crystal — the Hirota lattice model — has been discussed.

1. Using the superposition formula one can construct the multisoliton, breather and wobbling kink solutions. The nonlinear superposition formula is the recurrence relation allowing to construct more complicated soliton solutions using more simple soliton solutions. The «breather-breather» and «kink-breather» solutions describe the processes of pair collision of DBs with each other and with kinks (shock waves) respectively. The limiting cases of these solutions describe the scattering of the linear waves on the nonlinear excitations.

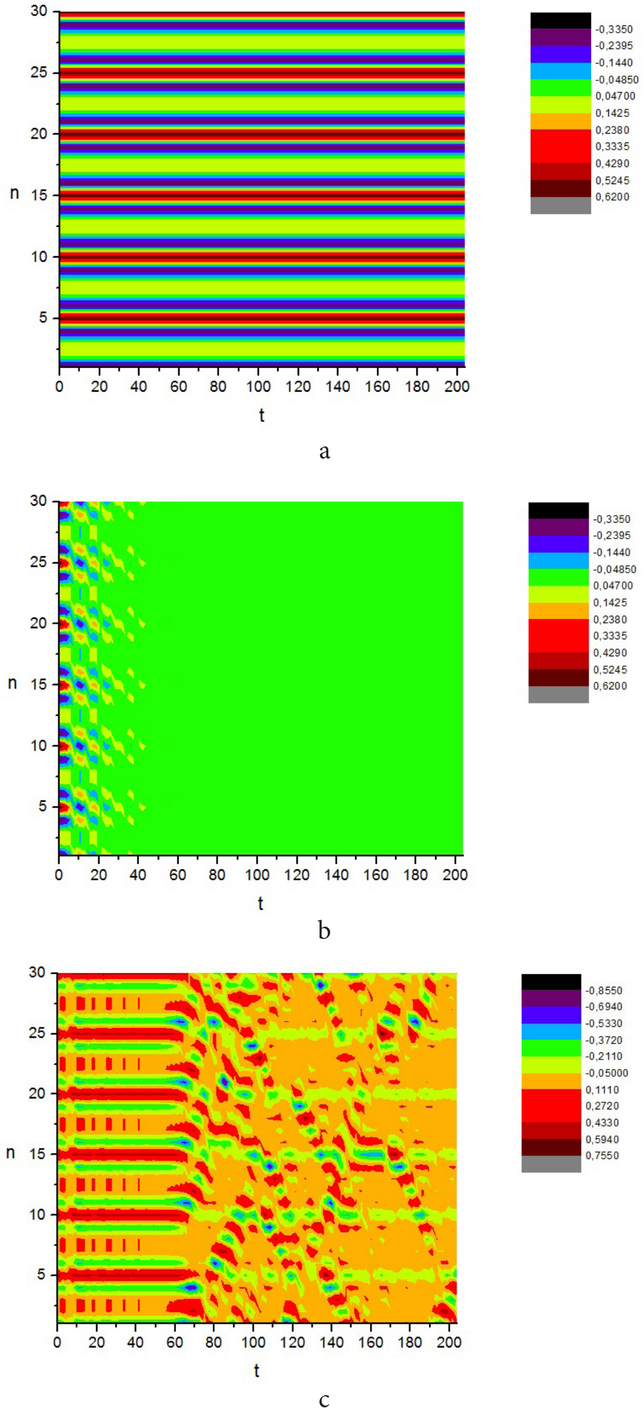

Fig. 3. A space time contour plot $\O_n(t)$. Initial condition is the breather superlattice solution of type II. N=30, M=6, χ=0.99. Time is measured in the periods of the oscillation (49). Boundary conditions are periodic. (a) The dynamics of the ideal lattice. λ=0, F_i=0. (b) The dynamics of the lattice with dissipation. λ=0.1, F_i=0. (c) The dynamics of the lattice with dissipation and point external periodic forces acting on the centers-of-mass of the each breather. λ=0.1, F_i=0.5cos(ωt), i=5,10,15,20,25,30. The frequency of the external force equals to the frequency of the breather lsuperattice solution of type II.

The analytic results for the scattering data — center-of-mass shifts of the DBs and shock waves as well as the shifts of phase oscillations of DBs and linear waves has been presented. It was shown that the one-parametric solitons and breathers as well as two breathers of the Hirota lattice model interact with each other by means of the effective short-range forces of attraction.

2. The developed Hamiltonian approach is used to describe the dynamics of the DBs and shock waves of the Hirota lattice model as particle-like excitations. Using the expression for the discrete analog of the field pulse for the nonlinear lattice systems the Hamiltonian functions for DBs and shock waves in terms of the collective coordinates have been obtained. The Hirota lattice model has been generalized by considering the terms corresponding to the external forces and dissipation processes.

3. In the long-wave limit the Hirota lattice model reduces to the exactly integrable modified Korteweg — de Vries (mKdV) equation. The breather solution of the mKdV equation and its quasiclassical spectrum has been discussed. Highly localized states in the chain can be qualitatively described by considering the model of the Hirota anharmonic oscillator. The solution describing the oscillations of the Hirota oscillator, the dependence of frequency on energy and the energy spectrum has been found. It was shown that using the nonlinear change of variables the model of the Hirota oscillator can be reduced to the well-known model of the Duffing oscillator.

4. The exact solutions in the form of the nonlinear spatial periodic waves in terms of the Jacobi elliptic functions have been analyzed. These solutions describe the discrete breathers and shock waves superlattices. The periodic and zero-fixed boundary conditions have been considered for the arbitrary number of sites. The analog of the DB for the finite-size lattices has been presented. It was shown that the breather solution is energetically more favorable then the corresponding homogeneous solution with the same frequency.

5. Numerical experiments show that the breather superlattices are dynamically metastable. The lifetime of these dynamical structures is much longer then the period of oscillations. The shock waves superlattices remained stable in the ideal lattice during the time of the numerical experiment. It was found that the lifetime of breather superlattices in the dissipative lattice can be extended through the concurrent application of ac driving and viscous damping terms.

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